Hw 7 - 2050B

1. (Terminology of N'd (= Neighbourhood). For $c \in \mathbb{R}$, by a n'd U of c we mean that $U \supseteq V_\delta(c)$ for some $\delta > 0$. For convenience, we can also call a set $U \subseteq \mathbb{R}$, a n'd of $+\infty$ if $\exists$ finite $\gamma \in \mathbb{R}$ such that $U \supseteq V_\gamma(+\infty)$, where

$$V_\gamma(+\infty) \stackrel{\text{def}}{:=} \{ x \in \mathbb{R} : x > \gamma \} \quad (\subseteq \mathbb{R}).$$

(not include $\pm\infty$)

Similarly one defines $V_\gamma(-\infty)$ and n'd of $V_\gamma(-\infty)$. Using the above terminologies, show that for $\emptyset \neq A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$ that

(i) $c \in A^c \iff$ each n'd of c intersects $A \setminus \{c\}$.

Modelling this, we define for $\emptyset \neq A \subseteq \mathbb{R}$ that

$+\infty \in A^c$ to mean that each n'd of $+\infty$ intersects A $(= A \setminus \{+\infty\})$

$-\infty \in A^c$ - - - - - - - - - - $-\infty$ - - - - - - -

Show that

(ii) Let $c \in [-\infty, \infty]$, and let $U_1, U_2, \cdots U_n$ $(n \in \mathbb{N})$ be n'ds (= neighbourhoods) of c. Then $\bigcap_{i=1}^{n} U_i$ is a n'd of c

(iii). not true for infinitely many n'ds in the above (ii)

2. Let $f : A \to \mathbb{R}$ and $c \in A^c$ $(\subseteq [-\infty, \infty])$ and let $l \in [-\infty, \infty]$. Show that the definitions (given in Bartle, Lectures, Tutorials)

for $\displaystyle \lim_{\substack{x \to c \\ x \in A}} f(x) = \begin{cases} +\infty \\ l \in \mathbb{R} \\ -\infty \end{cases}$ $\left(\begin{array}{c} c \in \mathbb{R}, \text{ or } c = +\infty \text{ or} \\ c = -\infty \end{array} \right)$

are consistent with the following :

$$\lim_{\substack{x \to c \\ x \in A}} f(x) = \bar{l}$$

$\Leftrightarrow$ $\forall$ n'd U of $\bar{l}$, $\exists$ n'd W of c such that the image $\{f(w) : w \in W \cap (A \setminus \{c\})\} \subseteq U$

3*. Let $f, g : X \to (0, +\infty)$ and $x_0 \in X^c \cap \mathbb{R}$ $(X \subseteq \mathbb{R})$. Show that

(i) $\lim\limits_{\substack{x \to x_0 \\ x \in X}} f(x) = l \in [-\infty, \infty]$ iff $\lim\limits_{\substack{x \to x_0 \\ x \in X}} (-f(x)) = -l$

(ii) $\lim\limits_{\substack{x \to x_0 \\ x \in X}} g(x) = 0$ iff $\lim\limits_{\substack{x \to x_0 \\ x \in X}} \left(\frac{1}{g(x)}\right) = +\infty$

(can the assumption $g(x) \in (0, +\infty)$ $\forall x$ replaced by $g(x) \in \mathbb{R} \setminus \{0\}$ $\forall x$?)

(iii) If $\lim\limits_{\substack{x \to x_0 \\ x \in X}} f(x) = l \in (0, +\infty)$ and $\lim\limits_{\substack{x \to x_0 \\ x \in X}} g(x) = 0$ then

$$\lim_{\substack{x \to x_0 \\ x \in X}} \left(\frac{f(x)}{g(x)}\right) = +\infty$$

(iv) Show that $\lim\limits_{\substack{x \to 1 \\ x > 1}} \left(\frac{x}{x - \sqrt{x}}\right) = +\infty$ by two methods:

(a) Use the results (i)–(iii)

(b) Check from def.

4. Can the assumption $x_0 \in X^c \cap \mathbb{R}$ in Q3 relaxed to $x_0 \in X^c$ $(\subseteq [-\infty, \infty])$?

5. Do Q3, 4, 5, 9, 13 of §4.3 in Bartle (Star-Questions: Q13 and Q5(a), (b), (c))